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Probabilities and path-integral realization of exclusion statistics

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ABSTRACT

A microscopic formulation of Haldane's exclusions statistics is given in terms of a priori occupation probabilities of states. It is shown that negative probabilities are always necessary to reproduce fractional statistics. Based on this formulation, a path-integral realization for systems with exclusion statistics is derived. This has the advantage of being generalizable to interacting systems, and can be used as the starting point for further generalizations of statistics. As a byproduct, the vanishing of the heat capacity at zero temperature for exclusion statistics systems is proved.

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Statistics is an inherently quantum mechanical property of identical particles which, as the name suggests, modifies the statistical mechanical properties of large collections of such particles. It enters through symmetry properties of the wave-functions of many-body states, and this was the starting point, until some time ago, for various generalizations away from the standard bosonic and fermionic cases, such as parastatistics and anyons. Haldane, however, taking the term statistics more literally, defined a generalized *exclusion* statistics through the reduction of the Hilbert space of additional particles in a system due to the ones already present in the system [1]. He proposed then the definition

$$g = -\frac{\Delta d}{\Delta N} \tag{1}$$

where N is the number of particles in the system, d is the dimensionality of the single-particle Hilbert space, obtained by holding the quantum numbers of $N-1$ particles fixed, and Δd and ΔN are their variation keeping the size and boundary conditions of the system fixed. $g=0$ corresponds to bosons (no exclusion) while $g=1$ corresponds to fermions, excluding a single state for the remaining particles, the one they occupy.

On the basis of (1), Haldane proposed the combinatorial formula for the number of many-body states of N particles occupying a group of K states

$$M = \frac{[K-(g-1)(N-1)]!}{N![K-g(N-1)-1]!} \tag{2}$$

Based on the above, Wu derived the thermodynamical properties of particles with exclusion statistics [2] (see also [3,4]), and this system has received a lot of recent attention [5,6,7]. Appropriate generalizations for several species of identical particles also exist.

It is obvious from (1) and (2) that exclusions statistics makes sense only in a statistical sence, since Δd and M can become fractional for $\Delta N = 1$ or $N = 1$. It is, nevertheless, useful to attempt a *microscopic* realization and interpretation of

fractional exclusion statistics, and see what it implies for the one-particle states, and this is what will be done in this paper. Such a description has the obvious advantage of being generalizable to *interacting* particles, for which the notion of d becomes hard to define.

The starting point will be the grand partition function for exclusion-statistics particles ("g-ons," as called in [6]) in K states

$$Z(K) = \sum_{N=0}^{\infty} M(K,N) x^N \tag{3}$$

where we put $x = \exp(\mu - \varepsilon)/kT$ with μ the chemical potential and assumed that all K states are at the same energy ε. In the statistical limit of large K, $Z(K)$ should be extensive. This intruduces, then, the notion of a microscopic description of the system in which the above Z is the K-th power of a single-state partition function. Each single level can be occupied by any number of particles, but with an *a priori probability* P_n for each occupancy n independent of the temperature. We thus demand

$$Z(K,x) = \left(\sum_n P_n(K) x^n \right)^K = \sum_{N=0}^{\infty} M(K,N) x^N \tag{4}$$

for all x. The above probabilities must, in general, depend on K in order to satisfy (4). This reflects the fact that g-ons are not well-defined for microscopic systems (see also the remarks in [6]). If, however, $P_n(K)$ assume some (finite) asymptotic values as K goes to infinity (as they should for an extensive Z), then the above microscopic partition function becomes an accurate description in the statistical limit. To calculate P_n in this limit, we first notice that the combinatorial formula (2) counts (at least for integer g) the ways of placing N identical particles in K sites arranged into a one-dimensional open lattice, under the restriction that any two particles be *at least* g sites apart. Clearly, for large K, the "boundary condition" that the lattice is open cannot influence the statistical mechanics of the system.

We choose, then, to examine instead particles placed on a *periodic* lattice under the same restriction. This modifies the combinatorics into

$$M' = \frac{K[K-(g-1)N-1]!}{N!(K-gN)!} \tag{5}$$

Clearly (5) reproduces the standard bosonic and fermionic results for $g = 0, 1$. Repeating the analysis of [2], it can be verified that M' leads indeed to the same statistical mechanics as M. It is now pointed out that the P_n defined in terms of M' are *independent* of K. (Proof: Putting $P_n(K) = P_n + \mathcal{O}(1/K)$, where P_n are the asymptotic values at $K = \infty$, and equating terms of n-th order in x we obtain

$$KP_n(K) + K(K-1)P_{n-1}(K)P_1(K) + \cdots = M'(K, n) \tag{6}$$

Assuming that all $P_m(K)$ for $m < n$ are independent of K, all the terms in (6) other than $KP_n(K)$ are polynomials in K without constant term, and so is $M'(K, n)$. Therefore $P_n(K)$ cannot contain any terms of $\mathcal{O}(1/K)$. Since $P_1 = 1$, we inductively showed that all P_n are K-independent.) Therefore, P_n can be calculated from $M'(1, n)$, and we obtain

$$P_n = \prod_{m=2}^{n}\left(1 - \frac{gn}{m}\right) \tag{7}$$

It can be a postoriori checked that the expressions for $P_n(K)$ obtained from M converge to (7) for $K = \infty$. The above P_n for $n = 0, \ldots, 5$ and $g = \frac{1}{2}$ agree with the values calculated in [6] using a different approach.

The most obvious feature of the above expressions is that, unless $g = 0, 1$, they *always* become negative for some values of n. Therefore, their interpretation as probabilities is problematic. This is an inherent problem of fractional g-on statistics which cannot be rectified by, e.g., truncating $M(K, N)$ to zero for $N > K/g$. The description of the *statistical* system in terms of effective negative microscopic probabilities is, nevertheless, accurate and useful. Note, also, that the above P_n never truncate to zero for n above some maximal value (unless $g = 1$), unlike parafermions.

From the above, the single-level partition function $Z(x) \equiv Z$ can be shown to satisfy

$$Z^g - Z^{g-1} = x \tag{8}$$

This is a transcedental equation which in principle determines Z and whose power-series solution reproduces the P_n as coefficients. The average occupation number $\bar{n}$ is expressed as

$$\bar{n} = \frac{1}{Z} x \partial_x Z = x \partial_x W \tag{9}$$

where $W = \ln Z$ is the free energy (over $-kT$). It can be shown that (9) together with (8) imply for $\bar{n}$

$$(1 - g\bar{n})^g [1 - (g-1)\bar{n}]^{1-g} = \bar{n} x^{-1} \tag{10}$$

in accordance with the result of [2,3,4].

One immediate consequence of the above relations is the vanishing of the zero-temperature heat capacity of a g-on system C_0, defined as

$$C_0 = \int_0^\infty (\epsilon - \mu) d\beta [\bar{n}(\beta) + \bar{n}(-\beta) - \frac{1}{g}] \tag{11}$$

where $\beta = 1/kT$, and $1/g$ is the saturation density for n at zero temperature and $\varepsilon < \mu$. Using (9) we can express C_0 as

$$C_0 = W(x = \Lambda) - \frac{1}{g} \ln \Lambda - W(x = 0) \tag{12}$$

where Λ is a cutoff to be taken to infinity. From (8) we can deduce that $W(x = 0) = 0$ and $W(x = \Lambda) = \frac{1}{g} \ln \Lambda + \mathcal{O}(\Lambda^{-1/g})$. Therefore $C_0 = 0$ for all g, as conjectured in [6]. This expresses the fact that the ground state of the many-body g-on system is nondegenerate. This is expected as a generic feature of particle systems, but is explicitly verified here for g-ons.

It is easy to derive a duality relation for Z:

$$Z^{-1}(g, x^{-g}) + Z^{-1}(\frac{1}{g}, x) = 1 \tag{13}$$

From the above relation and (8), (9), the duality relation for the density is recovered [4,6]

$$g\bar{n}(g, x) + \frac{1}{g}\bar{n}(\frac{1}{g}, x^{-1/g}) = 1 \tag{14}$$

We regard the formula (13) as more fundamental since it seems to be more generic. For instance, parafermions of order $p = 1/g$ are defined such that at most p particles can be put per state with probabilities 1. Thus

$$Z_{par} = 1 + x + \cdots x^p = \frac{1 - x^{p+1}}{1 - x} \tag{15}$$

from which we can write the generalized parafermionic partition function $Z_{par}(g, x)$ by simply putting $p = 1/g$ above. It can be seen that Z_{par} also satisfies (13), but not (14).

The free energy W can be expressed as a power series in x

$$W = \sum_{n=1}^{\infty} \frac{w_n}{n} x^n \tag{16}$$

in terms of the "connected" weights

$$w_1 = P1 \ , \quad w_2 = 2P_2 - P_1^2 \ , \quad w_3 = 3P_3 - 3P_1P_2 + P_1^3 \tag{17}$$

etc. We find for w_n:

$$w_n = \prod_{m=1}^{n-1} \Big(1 - \frac{gn}{m}\Big) \tag{18}$$

These are remarkably similar to P_n (except for the range of m). Notice that the w_n are *not* probabilities, but rather virial coefficients. In fact, $w_n = 1$ for bosons and $w_n = (-)^{n-1}$ for fermions. Also, $w_2 = 1 - 2g$ [5].

From the above expressions for w_n we can find a path integral representation for the partition function of g-ons in an arbitrary external potential. We start from the usual euclidean path integral with periodic time β for N particles with action the sum of N one-particle actions, and sum over all particle numbers N with appropriate chemical potential weights. Since the particles are identical, we must also sum over paths where particles have exchanged final positions, with weights equal to the inverse symmetry factors of the parmutation to avoid overcounting (compare with Feynman diagrams). Thus the path integral for each N decomposes into sectors labeled by the elements of the permutation group $Perm(N)$. By the usual argument, the free energy will be given by the sum of all connected path integrals. It is obvious that these are the ones where the final positions of the particles are a cyclic permutation of the original ones (since these are the only elements of $Perm(N)$ that cannot be written as a product of commuting elements). These have a symmetry factor of $1/N$ corresponding to cyclic relabelings of particle coordinates (compare with the factors of $1/n$ included in (16)). They really correspond to one particle wrapping N times around euclidean time. Thus, if we weight these configurations with the extra factors w_N, as we have the right to do since they belong to topologically distinct sectors, we will reproduce the free energy of a distribution of g-ons on the energy levels of the one-body problem, that is

$$\mathcal{W}(\beta, \mu) = \sum_{N=1}^{\infty} e^{\mu N} \frac{1}{N} \int w_N \prod_{n=1}^{N} Dx_n(t_n) e^{-S_E[x_n(t_n)]} \tag{19}$$

where S_E is the one-particle euclidean action and the paths obey the boundary conditions $x_n(\beta) = x_{n+1}(0)$, $x_N(\beta) = x_0(0)$. (x can be in arbitrary dimensions.) The partition function will be the path integral over all disconnected components, with appropriate symmetry factors and a factor of w_n for each disconnected n-particle component.

It is clear that the above path integral is not unitary, since the weights w_n are not phases, nor does it respect cluster decomposition, since the w_n do not provide true representations of the permutation group (unlike the $g = 0, 1$ cases). This is

again a manifestation of the non-microscopic nature of exclusion statistics. It does make sense, nevertheless, at the statistical limit.

The above path-integral realization can be extended to other statistics. E.g., for parafermions of order p (with p integer) the corresponding weigts w_n are

$$w_n = -p \quad \text{for} \quad n = 0 \ \mathrm{mod}(p+1) \ , \qquad 1 \quad \text{otherwise.} \tag{20}$$

This representation is more economical than the one calling for p distinct flavors of fermions and projecting all states transforming in an irreducible representation of $Perm(p)$ into a unique quantum state. The origin of the apparent non-unitarity and breakdown of cluster decomposition in the above integral for parafermions is clear: it is due to the above projection, which must be inserted in the (unitary) many-flavor path integral.

The possibility to define statistics throught the choice of the coefficients w_n suggests other possible generalizations. Perhaps the simplest one is to choose

$$w_n = (-\alpha)^{n-1} \tag{21}$$

that is, one factor of $-\alpha$ for each unavoidable particle crossing. This leads to the statistical distribution for the average occupation number $\bar{n}$

$$\bar{n} = \frac{1}{e^{(\varepsilon-\mu)\beta} + \alpha} \tag{22}$$

which is the simplest imaginable generalization of the Fermi and Bose distribution. The combinatorial formula for putting N particles in K states for the above α-statistics is

$$M = \alpha^N \frac{(\frac{K}{\alpha})!}{N!(\frac{K}{\alpha} - N)!} = \frac{K(K-\alpha)(K-2\alpha)\cdots(K-(N-1)\alpha)}{N!} \tag{23}$$

This can be thought as a different realization of the exclusion statistics idea: the first particle put in the system has K states to choose, the next has $K - \alpha$ due to

the presence of the previous one an so on, and dividing by $N!$ avoids overcounting. Fermions and bosons correspond to $\alpha = 1$ and $\alpha = -1$ respectively, while $\alpha = 0$ corresponds to Boltzmann statistics (as is also clear from the path integral, in which no configurations where particles have exchanged positions are allowed, but factors of $1/N!$ are still included). The corresponding single-level probabilities are

$$P_n = \prod_{m=1}^{n-1} \frac{1 - m\alpha}{1 + m} \tag{24}$$

For $\alpha = 1/p$ with p integer (a fraction of a fermion), the above probabilities are all positive for n up to p and vanish beyond that. For $\alpha < 0$ all probabilities are positive and nonzero. Thus, the above system has a bosonic ($\alpha < 0$) and a fermionic ($\alpha > 0$) sector, with Boltzmann statistics as the separator. It is a plausible alternative definition of exclusion statistics, due to (23), and has many appealing features, not shared by the standard (Haldane) exclusion statistics, such as positive probabilities, a maximum single-level occupancy in accordance with the fraction of a fermion that α represents, and analytic expressions for all thermodynamic quantities. It would be interesting to find a physical system in which these statistics are realized.

We conclude by pointing out that, once we have the path integral (19) we can easily extend the notion of exclusion statistics to interacting particles: we simply replace the action $\sum_n S_E[x_n]$ by the full interacting N-particle action, thus circumventing all difficulties with combinatorial formulae. In the interacting case one has to work with the full partition function, rather than the free energy (19), since topologically disconnected diagrams are still dynamically connected through the interactions and do not factorize. Applications of the above on physical systems, as well as possible generalizations to the many-flavor mutual-statistics case are left for future work.

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REFERENCES

1. F.D.M. Haldane, *Phys. Rev. Lett.* **67** (1991) 937.
2. Y.-S. Wu, *Phys. Rev. Lett.* **73** (1994) 922.
3. S. B. Isakov, *Int. Jour. Mod. Phys.* **A9** (1994) 2563.
4. A.K. Rajakopal, *Phys. Rev. Lett.* **74** (1995) 1048.
5. M.V.N. Murthy and R. Shankar, *Phys. Rev. Lett.* **72** (1994)
6. C. Nayak and F. Wilczek, PUPT-1466, IASSNS-94/25, to appear in Physical Review Letters.
7. D. Karabali and V.P. Nair, IASSNS-HEP-94/88, CCNY-HEP-94/9, to appear in Nuclear Physics B [FS].